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Reduced-Order Compensation Using the Hyland–Bernstein Optimal Projection Equations

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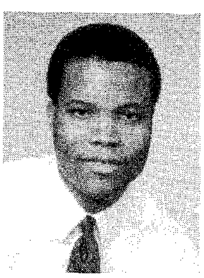
Gradient-based homotopy algorithms have previously been developed for synthesizing H_2 optimal reduced-order dynamic compensators. These algorithms are made efficient and avoid high-order singularities along the homotopy path by constraining the controller realization to a minimal parameter basis. The resultant homotopy algorithms, however, sometimes experience numerical ill conditioning or failure due to the minimal parameterization constraint. A new homotopy algorithm is presented that is based on solving the optimal projection equations, a set of coupled Riccati and Lyapunov equations that characterize the optimal reduced-order dynamic compensator. Path following in the proposed algorithm is accomplished using a predictor/corrector scheme that computes the prediction and correction steps by efficiently solving a set of four Lyapunov equations coupled by relatively low-rank linear operators. The algorithm does not suffer from ill conditioning because of constraining the controller basis and often exhibits better numerical properties than the gradient-based homotopy algorithms. The performance of the algorithm is illustrated by considering reduced-order control design for the benchmark four disk axial vibration problem and also reduced-order control of the Active Control Technique Evaluation for Spacecraft structure.



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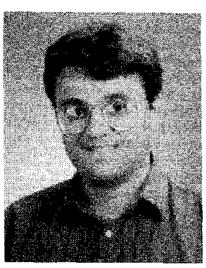

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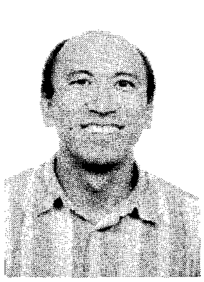

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Nomenclature

$\mathbb{E}, \boldsymbol{R}^n, \boldsymbol{R}^{m\times n}$ = expected value, $n \times 1$ real vectors, $m \times n$ real matrices
I_r, tr X = $r \times r$ identity matrix, trace of square matrix X
vec $(\cdot)$ = column stacking operator
$X^\dagger$, $X^\#$ = Moore–Penrose generalized inverse, group inverse of matrix X (Ref. 27)
$\|X\|_F^2$, $\|X\|_A$ = Frobenius norm $\triangleq$ tr XX^T, absolute norm $\triangleq \max_{i,j} |X_{i,j}|$
x_{ij} or $X_{i,j}$ = (i, j) element of matrix X
$Y \geq X, Y > X$ = $Y - X$ is nonnegative definite, $Y - X$ is positive definite

I. Introduction

THE design of reduced-order dynamic compensators is of practical importance because of limitations on the throughput of control processors. Hence, an important research area has involved the development of techniques for synthesizing H_2 optimal reduced-order compensators. Most of the techniques for designing optimal reduced-order compensators have been gradient-based parameter optimization methods that represent the controller by some parameter vector and attempt to find a vector for which the gradient of the performance index is zero, or, equivalently, the cost functional is minimal.

In the survey paper by Makila and Toivonen,[1] several gradient-based approaches were discussed. Levine–Athans type algorithms[2–7] are based on using some standard optimization methods (e.g., conjugate gradient algorithms) to iteratively solve the necessary conditions of optimality that minimize the cost increment. This approach requires the solution of a nonlinear matrix equation at each correction step but guarantees a cost descent direction without a line search. The Anderson–Moore algorithm[8] is based on minimizing a quadratic, positive-definite approximation of the second-order Taylor series expansion of the cost function increment. The descent Anderson–Moore approach utilizes gradient search schemes to guarantee the cost is reduced at each iteration and enhance convergence to a stationary point of the cost function.[9,10] For Newton-like approaches,[11] instead of approximating the Hessian of the cost functional with a positive-definite matrix, the actual second-order expansion is minimized, which involves computing the Newton correction step as the solution of a system of linear matrix equations at each iteration.

Recently, homotopy algorithms have been developed for the synthesis of optimal reduced-order compensators.[12–15] A gradient-based algorithm has been developed[15] that is made efficient and avoids high-order singularities along the homotopy path by constraining the controller realization to a minimal parameter basis. These algorithms[15] sometimes exhibit numerical ill conditioning or can even fail because of the basis constraint. This is because minimal parameterizations of a given form may not exist at each point along the homotopy path or may force the algorithm to be ill conditioned when the transformation to the given basis is ill conditioned. Nonminimal parameterizations exhibit singularities along the homotopy path that can be handled heuristically but may also lead to ill conditioning. This ill conditioning is also observed outside of the context of homotopy algorithms by Kuhn and Schmidt.[16] Similar conclusions are presented in Refs. 17 and 18 for the closely related H_2 optimal model reduction problem.

The homotopy algorithm of Ref. 19 was based on solving the optimal projection equations developed by Hyland and Bernstein.[20] The optimal projection equations are a set of coupled Riccati and Lyapunov equations that characterize extremal points to the optimal H_2 reduced-order dynamic compensation problem. The equations decouple and the Riccati equations specialize to the standard linear quadratic Gaussian (LQG) Riccati equations when the compensator is constrained to be full order. The initial homotopy algorithm[19] for solving the optimal projection equations utilized a very crude path following scheme in which the Riccati equations and Lyapunov equations were not updated simultaneously. This caused the algorithm to exhibit poor convergence properties, especially as the control authority was increased.

This paper presents a homotopy algorithm to solve the optimal projection equations that simultaneously updates the coupled Riccati and Lyapunov equations. The path following is accomplished using a predictor/corrector integration scheme that computes the prediction and correction steps by solving a set of four Lyapunov equations coupled by relatively low-rank linear operators. These equations are solved efficiently by using the technique presented in Ref. 21. This helps to avoid the very large dimensionality of similar algorithms based on the optimal projection equations for H_2 model reduction.[22,23] A model reduction algorithm that uses a similar approach to that used here is found in Ref. 24. Also, a related algorithm for full-order maximum entropy robust design is presented in Ref. 25. These results all show that algorithms based on the optimal projection equations tend to avoid the numerical ill conditioning experienced in gradient-based algorithms arising from constraints on the realization of the reduced-order model or controller.

The current homotopy algorithm, unlike some of the previous algorithms,[17,18,22,23] assumes that the homotopy curve is monotonic with respect to the homotopy parameter. As discussed in Ref. 13, this assumption may not always be satisfied. It appears to be possible to extend the algorithm to relax this assumption without significantly increasing the required computations by using a technique related to that developed in Ref. 26. This, however, is a subject of future research.

The paper is organized as follows. Section II presents the optimal projection equations for the H_2 reduced-order control problem. Section III gives a brief synopsis of homotopy methods. Next, Sec. IV develops a new homotopy algorithm for optimal reduced-order controller design based on the optimal projection equations. Section V illustrates the algorithm with two illustrative examples. Finally, Sec. VI presents the conclusion.

II. H_2 Optimal Reduced-Order Dynamic Compensation

Consider the nth-order linear time-invariant plant

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1 w(t) \tag{1}$$

$$y(t) = Cx(t) + Du(t) + D_2 w(t) \tag{2}$$

where (A, B) is stabilizable, (A, C) is detectable, $x \in \boldsymbol{R}^n$, $u \in \boldsymbol{R}^m$, $y \in \boldsymbol{R}^l$, and $w \in \boldsymbol{R}^d$ is a standard white noise disturbance with intensity I_d and rank $D_2 = l$. The intensities of $D_1 w(t)$ and $D_2 w(t)$ are thus given, respectively, by $V_1 \triangleq D_1 D_1^T \geq 0$, and $V_2 \triangleq D_2 D_2^T > 0$. For convenience, we assume that $V_{12} \triangleq D_1 D_2^T = 0$, that is, the plant disturbance and measurement noise are uncorrelated. The goal of the optimal reduced-order dynamic compensation problem is to determine an n_cth order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \tag{3}$$

$$u(t) = -C_c x_c(t) \tag{4}$$

that satisfies the following two design criteria: 1) the closed-loop system corresponding to Eqs. (1–4) given by

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{D}w(t) \tag{5}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \qquad \tilde{A} \triangleq \begin{bmatrix} A & -BC_c \\ B_c C & A_c - B_c D C_c \end{bmatrix}, \qquad \tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_c D_2 \end{bmatrix} \tag{6}$$

is asymptotically stable; and 2) the steady-state quadratic performance criterion

$$J(A_c, B_c, C_c) \triangleq \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left[x^T(s)R_1 x(s) + u^T(s)R_2 u(s)\right] \mathrm{d}s \tag{7}$$

where $R_1 \geq 0$ and $R_2 > 0$, is minimized.

Although a cross-weighting term of the form $2x^T(t)R_{12}u(t)$ can also be included in Eq. (7), we shall not do so here to facilitate the presentation. With the first criterion, we restrict our attention to the set of stabilizing compensators, $\mathcal{S}_c \triangleq \{(A_c, B_c, C_c): \tilde{A}$ is asymptotically stable$\}$ that guarantees that the cost J is finite and independent of initial conditions. The cost (7) can now be expressed as

$$J(A_c, B_c, C_c) = \lim_{t\to\infty} \mathbb{E}[\tilde{x}^T(t)\tilde{R}\tilde{x}(t)] \tag{8}$$

where

$$\tilde{R} \triangleq \begin{bmatrix} R_1 & 0 \\ 0 & C_c^T R_2 C_c \end{bmatrix} \tag{9}$$

Next, by introducing the performance variables

$$z(t) \triangleq E_1 x(t) + E_2 u(t) = \tilde{E}\tilde{x}(t) \tag{10}$$

where $\tilde{E} \triangleq [E_1 \quad E_2 C_c]$, and defining the transfer function from disturbances w to performance variables z by

$$\tilde{H}(s) \triangleq \tilde{E}(sI_{\tilde{n}} - \tilde{A})^{-1}\tilde{D}$$

where $\tilde{n} \triangleq n + n_c$, it can be shown that when $\tilde{A}$ is asymptotically stable, Eq. (8) is given by

$$J(A_c, B_c, C_c) = \|\tilde{H}(s)\|_2^2 \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} \|\tilde{H}(j\omega)\|_F^2 \,\mathrm{d}\omega$$

For convenience we define the matrices $R_1 \triangleq E_1^T E_1$ and $R_2 \triangleq E_2^T E_2$ that are the H_2 weights for the state and control variables. Since $\tilde{A}$ is asymptotically stable, there exist nonnegative-definite matrices $\tilde{Q} \in \boldsymbol{R}^{\tilde{n}\times\tilde{n}}$ and $\tilde{P} \in \boldsymbol{R}^{\tilde{n}\times\tilde{n}}$ satisfying the closed-loop steady-state covariance equation and its dual, i.e.,

$$\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + V = 0 \tag{11}$$

$$\tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} + \tilde{R} = 0 \tag{12}$$

where

$$\tilde{V} \triangleq \begin{bmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^T \end{bmatrix}$$

The cost functional (7) can now be expressed as $J(A_c, B_c, C_c) = \operatorname{tr} \tilde{Q}\tilde{R} = \operatorname{tr} \tilde{P}\tilde{V}$.

Before presenting the main theorem we present a key lemma concerning nonnegative definite matrices and several definitions.

Lemma 2.1.[28] Suppose $\hat{Q} \in \boldsymbol{R}^{n\times n}$ and $\hat{P} \in \boldsymbol{R}^{n\times n}$ are symmetric and nonnegative-definite and rank $\hat{Q}\hat{P} = n_c$. Then, the following statements hold:

1) $\hat{Q}\hat{P}$ is diagonalizable and has nonnegative eigenvalues.

2) The $n \times n$ matrix $\tau \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#}$, is idempotent, i.e., τ is an oblique projection and rank $\tau = n_c$. Furthermore, there exists a nonsingular matrix $W \in \boldsymbol{R}^{n\times n}$ such that

$$\tau = W\begin{bmatrix} I_{n_c} & 0 \\ 0 & 0 \end{bmatrix} W^{-1}$$

3) There exist $G, \Gamma \in \boldsymbol{R}^{n_c\times n}$ and nonsingular $M \in \boldsymbol{R}^{n_c\times n_c}$ such that $\hat{Q}\hat{P} = G^T M\Gamma$, and $\Gamma G^T = I_{n_c}$.

4) If G, Γ, and M satisfy property 3 then rank G = rank Γ = rank $M = n_c$, $(\hat{Q}\hat{P})^{\#} = G^T M^{-1}\Gamma$, $\tau = G^T\Gamma$, $\tau G^T = G^T$, and $\Gamma\tau = \tau$.

5) The matrices G, Γ, and M satisfying property 3 are unique except for a change of basis in $\boldsymbol{R}^{n_c}$, i.e., if G', Γ', and M' also satisfy property 4, then there exists nonsingular $T_c \in \boldsymbol{R}^{n_c\times n_c}$ such that $G' = T_c^T G$, $\Gamma' = T_c^{-1}\Gamma$, $M' = T_c^{-1} M T_c$. Furthermore, all such M are diagonalizable with positive eigenvalues.

6) Finally, if rank $\hat{Q}$ = rank $\hat{P}$ = rank $\hat{Q}\hat{P} = n_c$, there exists a nonsingular transformation $W \in \boldsymbol{R}^{n\times n}$ such that

$$\hat{Q} = W\begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix} W^T \quad \text{and} \quad \hat{P} = W^{-T}\begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix} W^{-1}$$

where $\Omega \in \boldsymbol{R}^{n_c\times n_c}$ is diagonal and nonsingular. In addition, $\hat{Q} = \tau\hat{Q} = \hat{Q}\tau^T = \tau\hat{Q}\tau^T$, and $\hat{P} = \tau^T\hat{P} = \hat{P}\tau = \tau^T\hat{P}\tau$.

Definition 2.1. A triple (G, M, Γ) satisfying property 3 of Lemma 2.1 is a projective factorization of $\hat{Q}\hat{P}$.

Definition 2.2. A compensator (A_c, B_c, C_c) is an extremal of the optimal reduced-order dynamic compensation problem if it satisfies the first-order necessary conditions of optimality, i.e., $\partial J/\partial A_c = 0$, $\partial J/\partial B_c = 0$, $\partial J/\partial C_c = 0$, where $J(A_c, B_c, C_c)$ is defined by Eq. (7).

Definition 2.3. A compensator (A_c, B_c, C_c) is an admissible extremal of the optimal reduced-order dynamic compensation problem if it is an extremal and is also in $\mathcal{S}_c$, i.e., the closed-loop system is asymptotically stable.

Finally, for convenience in stating the main results we define $\bar{\Sigma} \triangleq C^T V_2^{-1} C$, and $\Sigma \triangleq B R_2^{-1} B^T$.

Theorem 2.1.[20] Suppose (A_c, B_c, C_c) is an admissible extremal of the optimal reduced-order dynamic compensation problem. Then, there exist $n \times n$ nonnegative-definite matrices P, Q, $\hat{P}$, and $\hat{Q}$ such that A_c, B_c, and C_c are given by

$$A_c = \Gamma\big(A - Q\bar{\Sigma} - \Sigma P + QC^T V_2^{-1} D R_2^{-1} B^T P\big)G^T \tag{13}$$

$$B_c = \Gamma Q C^T V_2^{-1}, \qquad C_c = R_2^{-1} B^T P G^T \tag{14}$$

for some projective factorization (G, M, Γ) of $\hat{Q}\hat{P}$ and such that the following conditions are satisfied:

$$A^T P + PA + R_1 - P\Sigma P + \tau_\perp^T P\Sigma P\tau_\perp = 0 \tag{15}$$

$$AQ + QA^T + V_1 - Q\bar{\Sigma}Q + \tau_\perp Q\bar{\Sigma}Q\tau_\perp^T = 0 \tag{16}$$

$$(A - Q\bar{\Sigma})^T\hat{P} + \hat{P}(A - Q\bar{\Sigma}) + P\Sigma P - \tau_\perp^T P\Sigma P\tau_\perp = 0 \tag{17}$$

$$(A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^T + Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^T = 0 \tag{18}$$

$$\operatorname{rank}\hat{Q} = \operatorname{rank}\hat{P} = \operatorname{rank}\hat{Q}\hat{P} = n_c \tag{19}$$

$$\tau = (\hat{Q}\hat{P})(\hat{Q}\hat{P})^{\#}, \qquad \tau_\perp \triangleq I_n - \tau \tag{20}$$

Furthermore, the extremal cost is given by $J(A_c, B_c, C_c) = \operatorname{tr}[PV_1 + Q(P\Sigma P - \tau_\perp^T P\Sigma P\tau_\perp)]$ or, equivalently, $J(A_c, B_c, C_c) = \operatorname{tr}[QR_1 + P(Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^T)]$. Conversely, if there exist $n \times n$ nonnegative-definite matrices P, Q, $\hat{P}$, and $\hat{Q}$ satisfying Eqs. (15–20) then the compensator (A_c, B_c, C_c) given by Eqs. (13) and (14) is an extremal of the optimal fixed-order dynamic compensation problem. Furthermore, $\tilde{A}$ is asymptotically stable if and only if $(\tilde{A}, \tilde{E})$ is detectable [or, equivalently, $(\tilde{A}, \tilde{D})$ is stabilizable].

Remark 2.1. Partitioning $\tilde{Q}$ and $\tilde{P}$ given by Eqs. (11) and (12), respectively, as

$$\tilde{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}, \qquad Q_1 \in \boldsymbol{R}^{n\times n}, \qquad Q_2 \in \boldsymbol{R}^{n_c\times n_c}$$

$$\tilde{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}, \qquad P_1 \in \boldsymbol{R}^{n\times n}, \qquad P_2 \in \boldsymbol{R}^{n_c\times n_c}$$

it follows from Ref. 20 that P, Q, $\hat{P}$ and $\hat{Q}$ given by Eqs. (15–20) can be expressed as $P = P_1 - P_{12}P_2^{-1}P_{12}^T$, $Q = Q_1 - Q_{12}Q_2^{-1}Q_{12}^T$, $\hat{P} = P_{12}P_2^{-1}P_{12}^T$, and $\hat{Q} = Q_{12}Q_2^{-1}Q_{12}^T$, respectively.

Theorem 2.1 shows that one can compute an optimal reduced-order controller by solving the set of coupled, modified Riccati and Lyapunov equations (15–18) subject to the rank condition constraints (19). One approach to find a solution of Eqs. (15–18) is based on homotopy methods.

III. Homotopy Methods for the Solution of Nonlinear Algebraic Equations

A homotopy is a continuous deformation of one function into another. Over the past several years, homotopy or continuation methods (whose mathematical basis is algebraic topology and differential topology[29]) have received significant attention in mathematics literature and have been applied successfully to several important problems.[30–35] Recently, engineering literature has also begun to recognize the utility of these methods for engineering applications.[36–45] This section provides a very brief description of homotopy methods for finding the solutions of nonlinear algebraic equations. The reader is referred to Refs. 35, 36, and 46 for additional details.

The basic problem is as follows. Given sets U and V contained in $\boldsymbol{R}^n$ and a mapping $F: U \rightarrow V$, find solutions $u \in U$ to satisfy $F(u) = 0$. Homotopy methods embed this problem in a larger problem. In particular, let $H: U \times [0, 1] \rightarrow \boldsymbol{R}^n$ be such that 1) $H(u, 1) = F(u)$; 2) there exists at least one known $u_0 \in \boldsymbol{R}^n$ that is a solution to $H(\cdot, 0) = 0$, i.e., $H(u_0, 0) = 0$; 3) there exists a continuous curve $[u(\lambda), \lambda]$ in $\boldsymbol{R}^n \times [0, 1]$ such that $H[u(\lambda), \lambda] = 0$, $\lambda \in [0, 1]$, with $[u(0), 0] = (u_0, 0)$; and 4) the curve $[u(\lambda), \lambda]$ is differentiable. A homotopy algorithm then constructs a procedure to compute the actual curve such that the initial solution $u(0)$ is transformed to a desired solution $u(1)$ satisfying $H[u(1), 1] = F[u(1)] = 0$.

Now, differentiating $H[u(\lambda), \lambda] = 0$ with respect to λ yields Davidenko's differential equation $(\partial H/\partial u)/(\mathrm{d}u/\mathrm{d}\lambda) + \partial H/\partial\lambda = 0$, which together with $u(0) = u_0$ defines an initial value problem. The desired solution $u(1)$ is then obtained by numerical integration from 0 to 1. Some numerical integration schemes are described in Refs. 35 and 46.

IV. Homotopy Algorithm for H_2 Optimal Reduced-Order Control

This section begins by introducing a homotopy map based on the optimal projection equations. The construction of the initial point is then discussed in detail. Finally, the actual homotopy algorithm is presented.

A. Homotopy Map

To define the homotopy map we assume that the plant matrices (A, B, C, D), the cost weighting matrices (R_1, R_2), and the disturbance matrices (V_1, V_2) are functions of the homotopy parameter $\lambda \in [0, 1]$. In particular, the following is assumed:

$$\begin{bmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix} = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} + \lambda \left(\begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix} - \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} \right)$$

Also, $R_1(\lambda) = R_{1,0} + \lambda(R_{1,f} - R_{1,0})$, $R_2(\lambda) = R_{2,0} + \lambda(R_{2,f} - R_{2,0})$, $V_1(\lambda) = V_{1,0} + \lambda(V_{1,f} - V_{1,0})$, and $V_2(\lambda) = V_{2,0} + \lambda(V_{2,f} - V_{2,0})$. Note that the assumptions imply that $A(0) = A_0$, $B(0) = B_0, \ldots, V_2(0) = V_{2,0}$, and that $A(1) = A_f$, $B(1) = B_f, \ldots, V_2(1) = V_{2,f}$. For notational simplification, we also define $\Sigma(\lambda) \triangleq B(\lambda)R_2^{-1}(\lambda)B^T(\lambda)$ and $\bar{\Sigma}(\lambda) \triangleq C^T(\lambda)V_2^{-1}(\lambda)C(\lambda)$.

The homotopy formulation $H[(P, Q, \hat{P}, \hat{Q}), \lambda] = 0$ is thus given by

$$A(\lambda)^T P(\lambda) + P(\lambda)A(\lambda) + R_1(\lambda) + \tau^T(\lambda)P(\lambda)\Sigma(\lambda)P(\lambda)\tau(\lambda) - \tau^T(\lambda)P(\lambda)\Sigma(\lambda)P(\lambda) - P(\lambda)\Sigma(\lambda)P(\lambda)\tau(\lambda) = 0 \tag{21}$$

$$A(\lambda)Q(\lambda) + Q(\lambda)A(\lambda)^T + V_1(\lambda) + \tau(\lambda)Q(\lambda)\bar{\Sigma}(\lambda)Q(\lambda)\tau^T(\lambda) - \tau(\lambda)Q(\lambda)\bar{\Sigma}(\lambda)Q(\lambda) - Q(\lambda)\bar{\Sigma}(\lambda)Q(\lambda)\tau^T(\lambda) = 0 \tag{22}$$

$$[A(\lambda) - Q(\lambda)\bar{\Sigma}(\lambda)]^T \hat{P}(\lambda) + \hat{P}(\lambda)[A(\lambda) - Q(\lambda)\bar{\Sigma}(\lambda)] - \tau^T(\tau)P(\lambda)\Sigma(\lambda)P(\lambda)\tau(\lambda) + \tau^T(\lambda)P(\lambda)\Sigma(\lambda)P(\lambda) + P(\lambda)\Sigma(\lambda)P(\lambda)\tau(\lambda) = 0 \tag{23}$$

$$[A(\lambda) - \Sigma(\lambda)P(\lambda)]\hat{Q}(\lambda) + \hat{Q}(\lambda)[A(\lambda) - \Sigma(\lambda)P(\lambda)]^T - \tau(\lambda)Q(\lambda)\bar{\Sigma}(\lambda)Q(\lambda)\tau^T(\lambda) + \tau(\lambda)Q(\lambda)\bar{\Sigma}(\lambda)Q(\lambda) + Q(\lambda)\bar{\Sigma}(\lambda)Q(\lambda)\tau^T(\lambda) = 0 \tag{24}$$

where

$$\operatorname{rank} \hat{Q}(\lambda) = \operatorname{rank} \hat{P}(\lambda) = \operatorname{rank} \hat{Q}(\lambda)\hat{P}(\lambda) = n_c \tag{25}$$

$$\tau(\lambda) = \hat{Q}(\lambda)\hat{P}(\lambda)[\hat{Q}(\lambda)\hat{P}(\lambda)]^{\#} \tag{26}$$

and $\lambda \in [0, 1]$.

B. Initial System Selection

Before describing the general logic and features of the homotopy algorithm for H_2 optimal reduced-order dynamic compensation, we first discuss the importance of the homotopy initialization and some guidelines for choosing the initial system matrices. It is assumed that the designer has supplied a set of system and weighting matrices, $S_f = (A_f, B_f, C_f, D_f, R_{1,f}, R_{2,f}, V_{1,f}, V_{2,f})$ describing the optimization problem whose solution is desired. In addition, it is assumed that the designer has chosen an initial set of related system matrices $S_0 = (A_0, B_0, C_0, D_0, R_{1,0}, R_{2,0}, V_{1,0}, V_{2,0})$ that has an easily obtained $(P_0, Q_0, \hat{P}_0, \hat{Q}_0)$ that is either a solution or a good approximation to the solution of the optimal projection equations corresponding to the initial system [i.e., Eqs. (21–26) with $\lambda = 0$].

Although, in general, homotopy methods ease the restriction that the starting point be close to some optimal of the optimization problem, the initial guess does affect the performance of the homotopy algorithm. For example, it is always possible to choose the initial system S_0 such that (A_0, B_0, C_0, D_0) is nonminimal with minimal dimension n_c. In this case, it is easy to show that corresponding LQG compensator has minimal dimension $n_r \leq n_c$ and will usually have minimal dimension $n_r = n_c$. In the latter case, $(A_{c,0}, B_{c,0}, C_{c,0})$ is chosen as a minimal realization of the LQG compensator. We have seen experimentally, however, that the corresponding homotopy can lead to failure of the homotopy algorithm. Similar observations have been made in Ref. 13. In particular, Ref. 13 shows that allowing the plant parameters to vary along the homotopy path can lead to the development of destabilizing controllers or path bifurcations.

The reason that this type of homotopy would cause problems is somewhat intuitive since for a given λ, say $\lambda_1 \in [0, 1]$, a controller $[A_c(\lambda_1), B_c(\lambda_1), C_c(\lambda_1)]$ that stabilizes the plant $[A(\lambda_1), B(\lambda_1), C(\lambda_1), D(\lambda_1)]$ may not stabilize the plant $[A(\lambda_2), B(\lambda_2), C(\lambda_2), D(\lambda_2)]$ for $\lambda_2 \neq \lambda_1$. Hence, we present ways of constructing the initial system S_0 that do not require the plant parameters (A, B, C, D) to vary along the homotopy path. In this case, a controller that stabilizes the plant at λ_1 will also stabilize the plant at $\lambda_2 > \lambda_1$. This argument in itself does not ensure that at every step along the homotopy algorithm the controller design remains stabilizing. This is a subject that requires further research. It should be mentioned that another advantage of a homotopy that varies only the performance weights (R_1, R_2, V_1, V_2) is that the optimal controller at each point is optimal with respect to the real nominal plant (A_f, B_f, C_f, D_f).

Now, we present two options for constructing S_0 as proposed in Ref. 15. Option 1, the first alternative, is to choose A_0 to be stable (e.g., if A_f is stable, let $A_0 = A_f$ or if A_f is unstable, let $A_0 = A_f - \sigma I$ where σ is sufficiently large to ensure stability of A_0) and, as elaborated in Ref. 47 to choose either $(R_{1,0}, V_{2,0})$ or $(V_{1,0}, R_{2,0})$, where $R_{1,0} \geq 0$, $V_{1,0} \geq 0$, $R_{2,0} > 0$, and $V_{2,0} > 0$, as given in the following (all other initial parameters are equal to their final values):

1) In a basis in which

$$A_0 = \begin{bmatrix} (A_0)_{11} & 0 \\ (A_0)_{21} & (A_0)_{22} \end{bmatrix}, \qquad (A_0)_{11} \in \boldsymbol{R}^{n_c \times n_c}$$

choose $R_{1,0}$ to be of the form

$$R_{1,0} = \begin{bmatrix} (R_{1,0})_{11} & 0 \\ 0 & 0 \end{bmatrix}, \qquad (R_{1,0})_{11} \in \boldsymbol{R}^{n_c \times n_c}$$

and for some positive scalar α choose $V_{2,0} = \alpha V_{2,f}$.

2) In a basis in which A_0 has the equality just given, choose $V_{1,0}$ to be of the form

$$V_{1,0} = \begin{bmatrix} (V_{1,0})_{11} & 0 \\ 0 & 0 \end{bmatrix}, \qquad (V_{1,0})_{11} \in \boldsymbol{R}^{n_c \times n_c}$$

and for some positive scalar α choose $R_{2,0} = \alpha R_{2,f}$. As discussed in Ref. 47, α appearing in bases 1 and 2 can always be chosen sufficiently large so that the corresponding LQG compensator is nearly nonminimal. In this case, $(A_{c,0}, B_{c,0}, C_{c,0})$ is easily obtained by reducing the LQG compensator to its (nearly) minimal realization using an appropriate technique such as balanced controller reduction.[48] Next, form the closed-loop system consisting of (A_0, B_0, C_0, D_0) and $(A_{c,0}, B_{c,0}, C_{c,0})$ and compute the initial guess P_0, Q_0, $\hat{P}_0$, and $\hat{Q}_0$, respectively, using Remark 2.1. Since $(A_{c,0}, B_{c,0}, C_{c,0})$ is a close approximation to the minimal realization of the corresponding nearly nonminimal LQG compensator, $(P_0, Q_0, \hat{P}_0, \hat{Q}_0)$ is a good approximation of the solution of the optimal projection equations corresponding to the initial system [i.e., Eqs. (21–26) with $\lambda = 0$].

Option 2, the second alternative (which does not require A_0 to be stable), is based on the following experimental observation. The initial system can be chosen to correspond to a low authority control problem, e.g., one can choose $R_{2,0} = \alpha R_{2,f}$ and $V_{2,0} = \beta V_{2,f}$, with α and β large and let all other initial system parameters equal their final values. In this case it has been observed that the reduced-order controller $(A_{c,r}, B_{c,r}, C_{c,r})$ obtained by suboptimal reduction of an LQG controller will often yield virtually the same cost as the LQG controller,[49] hence indicating that $(A_{c,r}, B_{c,r}, C_{c,r})$ may be nearly optimal. In this case, we choose $(A_{c,0}, B_{c,0}, C_{c,0}) = (A_{c,r}, B_{c,r}, C_{c,r})$. (Note that these observations are partially explained by the results in Ref. 47.) Then, follow the same procedure described in option 1 to form the closed-loop system and compute the initial guess $(P_0, Q_0, \hat{P}_0, \hat{Q}_0)$.

C. Derivative and Correction Equations

The homotopy presented next uses a predictor/corrector numerical integration scheme. The prediction step requires derivatives $[\dot{P}(\lambda), \dot{Q}(\lambda), \dot{\hat{P}}, \dot{\hat{Q}}(\lambda)]$, where $\dot{M} \triangleq \mathrm{d}M/\mathrm{d}\lambda$, whereas the correction step is based on using a Newton correction, denoted as $(\Delta P, \Delta Q, \Delta\hat{P}, \Delta\hat{Q})$. Before constructing the derivative and correction equations, we state the following useful properties about the derivatives of the contragredient transformation of $(\hat{Q}, \hat{P})$.

Using Lemma 2.1, Eqs. (25) and (26) imply

$$\hat{Q}(\lambda) = W(\lambda)\Lambda(\lambda)W^T(\lambda) = W_1(\lambda)\Omega(\lambda)W_1^T(\lambda) \tag{27}$$

$$\hat{P}(\lambda) = U^T(\lambda)\Lambda(\lambda)U(\lambda) = U_1(\lambda)\Omega(\lambda)U_1^T(\lambda) \tag{28}$$

and

$$\tau(\lambda) = W(\lambda)\begin{bmatrix} I_{n_c} & 0 \\ 0 & 0 \end{bmatrix}U(\lambda) = W_1(\lambda)U_1^T(\lambda) \tag{29}$$

where

$$W(\lambda) \triangleq [W_1(\lambda) \quad W_2(\lambda)], \qquad W_1(\lambda) \in \boldsymbol{R}^{n\times n_c}, \qquad W_2(\lambda) \in \boldsymbol{R}^{n\times(n-n_c)} \tag{30}$$

$$U(\lambda) \triangleq \begin{bmatrix} U_1^T(\lambda) \\ U_2^T(\lambda) \end{bmatrix}, \qquad U_1(\lambda) \in \boldsymbol{R}^{n\times n_c}, \qquad U_2(\lambda) \in \boldsymbol{R}^{n\times(n-n_c)} \tag{31}$$

$$U(\lambda) = W^{-1}(\lambda) \tag{32}$$

or, equivalently,

$$U(\lambda)W(\lambda) = I_n \tag{33}$$

$$\Lambda(\lambda) \triangleq \begin{bmatrix} \Omega(\lambda) & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Omega(\lambda) \in \boldsymbol{R}^{n_c\times n_c} \tag{34}$$

and $\Omega(\lambda)$ is diagonal and positive definite. For notational simplicity, we omit the argument λ in the subsequent equations.

The derivative equations, obtained by differentiating Eqs. (21–24) with respect to λ, are given by

$$A_P^T\dot{P} + \dot{P}A_P + R_P(\dot{P}, \dot{Q}, \dot{\hat{P}}, \dot{\hat{Q}}) + R_P^T(\dot{P}, \dot{Q}, \dot{\hat{P}}, \dot{\hat{Q}}) + V_P + V_P^T + \dot{R}_1 = 0 \tag{35}$$

$$A_Q\dot{Q} + \dot{Q}A_Q^T + R_Q(\dot{P}, \dot{Q}, \dot{\hat{P}}, \dot{\hat{Q}}) + R_Q^T(\dot{P}, \dot{Q}, \dot{\hat{P}}, \dot{\hat{Q}}) + V_Q + V_Q^T + \dot{V}_1 = 0 \tag{36}$$

$$A_w^T\dot{\hat{P}} + \dot{\hat{P}}A_w + R_{\hat{P}}(\dot{P}, \dot{Q}, \dot{\hat{P}}, \dot{\hat{Q}}) + R_{\hat{P}}^T(\dot{P}, \dot{Q}, \dot{\hat{P}}, \dot{\hat{Q}}) + V_{\hat{P}} + V_{\hat{P}}^T = 0 \tag{37}$$

$$A_u\dot{\hat{Q}} + \dot{\hat{Q}}A_u^T + R_{\hat{Q}}(\dot{P}, \dot{Q}, \dot{\hat{P}}, \dot{\hat{Q}}) + R_{\hat{Q}}^T(\dot{P}, \dot{Q}, \dot{\hat{P}}, \dot{\hat{Q}}) + V_{\hat{Q}} + V_{\hat{Q}}^T = 0 \tag{38}$$

where A_P, A_Q, A_w, A_u, $R_P(\dot{P}, \dot{Q}, \dot{\hat{P}}, \dot{\hat{Q}})$, $R_Q(\dot{P}, \dot{Q}, \dot{\hat{P}}, \dot{\hat{Q}})$, $R_{\hat{P}}(\dot{P}, \dot{Q}, \dot{\hat{P}}, \dot{\hat{Q}})$, $R_{\hat{Q}}(\dot{P}, \dot{Q}, \dot{\hat{P}}, \dot{\hat{Q}})$, V_P, V_Q, $V_{\hat{P}}$, and $V_{\hat{Q}}$ are defined in the Appendix.

The correction equations, derived similarly by using the relationship between the Newton's method and a particular error homotopy, are given by

$$A_P^T\Delta P + \Delta P A_P + R_P(\Delta P, \Delta Q, \Delta\hat{P}, \Delta\hat{Q}) + R_P^T(\Delta P, \Delta Q, \Delta\hat{P}, \Delta\hat{Q}) + E_P^* = 0 \tag{39}$$

$$A_Q\Delta Q + \Delta Q A_Q^T + R_Q(\Delta P, \Delta Q, \Delta\hat{P}, \Delta\hat{Q}) + R_Q^T(\Delta P, \Delta Q, \Delta\hat{P}, \Delta\hat{Q}) + E_Q^* = 0 \tag{40}$$

$$A_w^T\Delta\hat{P} + \Delta\hat{P}A_w + R_{\hat{P}}(\Delta P, \Delta Q, \Delta\hat{P}, \Delta\hat{Q}) + R_{\hat{P}}^T(\Delta P, \Delta Q, \Delta\hat{P}, \Delta\hat{Q}) + E_{\hat{P}}^* = 0 \tag{41}$$

$$A_u\Delta\hat{Q} + \Delta\hat{Q}A_u^T + R_{\hat{Q}}(\Delta P, \Delta Q, \Delta\hat{P}, \Delta\hat{Q}) + R_{\hat{Q}}^T(\Delta P, \Delta Q, \Delta\hat{P}, \Delta\hat{Q}) + E_{\hat{Q}}^* = 0 \tag{42}$$

where $R_P(\Delta P, \Delta Q, \Delta\hat{P}, \Delta\hat{Q})$, $R_Q(\Delta P, \Delta Q, \Delta\hat{P}, \Delta\hat{Q})$, $R_{\hat{P}}(\Delta P, \Delta Q, \Delta\hat{P}, \Delta\hat{Q})$, and $R_{\hat{Q}}(\Delta P, \Delta Q, \Delta\hat{P}, \Delta\hat{Q})$ are, again, defined in the Appendix and E_P^*, E_Q^*, $E_{\hat{P}}^*$ and $E_{\hat{Q}}^*$ are the errors in Eqs. (21–24) with $\lambda = \lambda^*$.

A detailed derivation of Eqs. (35–42) is given in the Appendix. Comparing Eqs. (39–42) with Eqs. (35–38) reveals that the derivative and correction equations are identical in form. Thus, only one solution procedure would be required to solve both sets of equations. Each set of equations consist of four coupled Lyapunov equations. Since these equations are linear, using Kronecker algebra,[50] they can be converted to the vector form $\mathcal{A}\chi = b$ where for Eqs. (39–42) χ is a vector containing the independent elements of ΔP, ΔQ, ΔW_1, ΔU_1, and $\Delta\Omega$. $\mathcal{A}$ is then a square matrix of dimension $n(n+1) + (2nn_c + n_c^2)$. Inversion of $\mathcal{A}$ is, hence, very computationally intensive for even relatively small problems (e.g., $n = 20$, $n_c = 10$).

Fortunately, the coupling terms $R_{\Delta P}$, $R_{\Delta Q}$, $R_{\Delta\hat{P}}$, and $R_{\Delta\hat{Q}}$, which are linear functions of $(\Delta P, \Delta Q, \Delta\hat{P}, \Delta\hat{Q})$ or, equivalently, $(\Delta P, \Delta Q, \Delta W_1, \Delta U_1, \Delta\Omega)$ in Eqs. (39–42), have relatively low ranks. Hence, the technique of Ref. 21, which exploits this low-rank property, can be used to efficiently solve Eqs. (39–42) [or, equivalently, Eqs. (35–38)]. In particular, this solution procedure requires inversion of a square matrix of dimension $(2n + n_c)(m + l) + 1$ and the solution of four sets of standard $n \times n$ Lyapunov equations, which has much less computational burden than the approach using Kronecker algebra as described in the preceding paragraph. In comparison, the dimension of the homotopy Jacobian inverted in the minimal parameterization approach is $n_c(m + l)$, which is smaller than the characteristic dimension associated with this approach. However, the algorithms based on these minimal parameter bases sometimes exhibit numerical ill conditioning or can even fail because of the basis constraint. The details of the solution procedure are described in Ref. 51.

Finally, note that if the homotopy path exists, the solution to the coupled Lyapunov equations will be well posed. Hence, the matrices A_P, A_Q, A_u, and A_w in Eqs. (35–42) will have the property that any two eigenvalues of a given matrix will not sum to zero.

D. Overview of the Homotopy Algorithm

In the following, we present an outline of the homotopy algorithm. This algorithm describes a predictor/corrector numerical integration scheme. To force the rank conditions (25) of $\hat{Q}$ and $\hat{P}$ during intermediate steps, we use the following scheme to update $(P, Q, \hat{P}, \hat{Q})$ along the homotopy path. First, using Eqs. (A25–A27) and (A45–A47) from the Appendix and the, algorithms described in Ref. 51, the prediction $(\dot{\hat{Q}}, \dot{\hat{P}})$ and correction $(\Delta\hat{Q}, \Delta\hat{P})$ are first converted to $(\dot{W}_1, \dot{U}_1, \dot{\Omega})$ and $(\Delta W_1, \Delta U_1, \Delta\Omega)$, respectively. Note that $\dot{\Omega}$ and $\Delta\Omega$ are forced to be $n_c \times n_c$ diagonal matrices with this formulation. Next, we update (P, Q, W_1, U_1, Ω) with these predictions/corrections. Finally, new $(\hat{Q}, \hat{P})$ are constructed with updated (W_1, U_1, Ω) using Eqs. (27) and (28) and the rank conditions (25) are maintained.

There are several options to be chosen initially. These options are enumerated before presenting the actual algorithm. Note that each option corresponds to a particular flag being assigned some integer value.

1. *Prediction Scheme Options*

Here we use the notation λ_0, λ_{-1}, and λ_1 representing the values of λ at the current point on the homotopy curve, the previous point, and the next point, respectively. Also, $\dot{M} \triangleq \mathrm{d}M/\mathrm{d}\lambda$ and $\theta(\lambda)$ is a vector representation of $[P(\lambda), Q(\lambda), W_1(\lambda), U_1(\lambda), \Omega(\lambda)]$. The predictions are as follows:

1) No prediction, pred = 0: this option assumes that $\theta(\lambda_1) = \theta(\lambda_0)$.

2) Linear prediction, pred = 1: this option assumes that $\theta(\lambda_1)$ is predicted using $\dot{\theta}(\lambda_0)$ and $\theta(\lambda_0)$. In particular, $\theta(\lambda_1) = \theta(\lambda_0) + (\lambda_1 - \lambda_0)\dot{\theta}(\lambda_0)$.

3) Cubic polynomial prediction, pred = 2: this prediction of $\theta(\lambda_1)$ requires $\theta(\lambda_0)$, $\dot{\theta}(\lambda_0)$, $\theta(\lambda_{-1})$ and $\dot{\theta}(\lambda_{-1})$. In particular, vec $[\theta(\lambda_1)] = a_0 + a_1\lambda_1 + a_2\lambda_1^2 + a_3\lambda_1^3$, where a_0, a_1, a_2, and a_3 are computed by solving

$$[a_0 \quad a_1 \quad a_2 \quad a_3]\begin{bmatrix} 1 & 0 & 1 & 0 \\ \lambda_{-1} & 1 & \lambda_0 & 1 \\ \lambda_{-1}^2 & 2\lambda_{-1} & \lambda_0^2 & 2\lambda_0 \\ \lambda_{-1}^3 & 3\lambda_{-1}^2 & \lambda_0^3 & 3\lambda_0^2 \end{bmatrix} = \begin{bmatrix} \mathrm{vec}\,[\theta(\lambda_{-1})] \\ \mathrm{vec}\,[\dot{\theta}(\lambda_{-1})] \\ \mathrm{vec}\,[\theta(\lambda_0)] \\ \mathrm{vec}\,[\dot{\theta}(\lambda_0)] \end{bmatrix}^T$$

Note that if $\theta(\lambda_{-1})$ and $\dot{\theta}(\lambda_{-1})$ are not available (as occurs at the initial iteration of the homotopy algorithm), then $\theta(\lambda_1)$ is predicted using the linear prediction (pred = 1).

2. *Basis Options for Solving the Coupled Lyapunov Equations*

The main computational burden of the algorithm given next is the solution of the four coupled modified Lyapunov equations (35–38) or (39–42) at each prediction step or correction iteration. Efficient solutions of these equations, as described in Ref. 51, makes the algorithm feasible for large-scale systems. The most desired solution procedure is based on diagonalizing the coefficient matrices A_P, A_Q, A_w, and A_u of the coupled Lyapunov equations. This is usually possible. However, it is also possible that this diagonalization will be intractable for some points along the homotopy path. A numerical conditioning test is embedded in the program to determine whether the coefficient matrices are truly diagonalizable. If they are not, then the coupled Lyapunov equations are solved using the Schur decomposition. A second option relies exclusively on the Schur decomposition:

A_P, A_Q, A_w, and A_u are diagonalized when solving Eqs. (35–38) or (39–42); basis = 1.

A_P, A_Q, A_w, and A_u are in Schur form when solving Eqs. (35–38) or (39–42); basis = 2.

3. *Outline of the Homotopy Algorithm*

Step 1) Initialize loop = 0, $\lambda = 0$, $\Delta\lambda \in (0, 1]$, $S = S_0$, $(P, Q, \hat{P}, \hat{Q}) = (P_0, Q_0, \hat{P}_0, \hat{Q}_0)$.

Step 2) Let loop = loop + 1. If loop = 1, then go to step 4.

Step 3) Advance the homotopy parameter λ and predict the corresponding $P(\lambda)$, $Q(\lambda)$, $\hat{P}(\lambda)$, and $\hat{Q}(\lambda)$ as follows:

3a) Let $\lambda_0 = \lambda$.

3b) Let $\lambda = \lambda_0 + \Delta\lambda$.

3c) If pred ≥ 1, then perform the next step to compute $\dot{P}(\lambda_0)$, $\dot{Q}(\lambda_0)$, $\dot{\hat{P}}(\lambda_0)$, and $\dot{\hat{Q}}(\lambda_0)$ according to Eqs. (35–38). Else, let $P(\lambda) = P(\lambda_0)$, $Q(\lambda) = Q(\lambda_0)$, $\hat{P}(\lambda) = \hat{P}(\lambda_0)$, and $\hat{Q}(\lambda) = \hat{Q}(\lambda_0)$ and go to step 4), i.e., no prediction is performed.

3d) Transform A_P, A_Q, A_w, and A_u into suitable matrix form according to the option defined by basis, then solve Eqs. (35–38) as described in Ref. 51.

3e) Compute $[\dot{W}_1(\lambda_0), \dot{U}_1(\lambda_0), \dot{\Omega}(\lambda_0)]$ from $[\dot{\hat{Q}}(\lambda_0), \dot{\hat{P}}(\lambda_0)]$ by using Eqs. (A25–A27) and the procedure described in Ref. 51.

3f) Predict $[P(\lambda), Q(\lambda), W_1(\lambda), U_1(\lambda), \Omega(\lambda)]$ by using the option defined by pred.

3g) Compute $[\hat{Q}(\lambda), \hat{P}(\lambda)]$ from $[W_1(\lambda), U_1(\lambda), \Omega(\lambda)]$ using Eqs. (27) and (28).

Step 4) Correct the current approximations $P(\lambda^*)$, $Q(\lambda^*)$, $\hat{P}(\lambda^*)$, and $\hat{Q}(\lambda^*)$ as follows:

4a) Compute the errors $(E_P^*, E_Q^*, E_{\hat{Q}}^*, E_{\hat{P}}^*)$ in the correction equations (A34–A37).

4b) Transform A_P, A_Q, A_w, and A_u into suitable matrix form by using the option defined by basis, then solve Eqs. (39–42) as described in Ref. 51 for ΔP, ΔQ, $\Delta\hat{P}$, and $\Delta\hat{Q}$.

4c) Compute $(\Delta W_1, \Delta U_1, \Delta\Omega)$, from $(\Delta\hat{Q}, \Delta\hat{P})$ by using Eqs. (A45–A47) and the algorithms described in Ref. 51.

4d) Let $P(\lambda) \leftarrow P(\lambda) + \Delta P$, $Q(\lambda) \leftarrow Q(\lambda) + \Delta Q$, $W_1(\lambda) \leftarrow W_1(\lambda) + \Delta W_1$, $U_1(\lambda) \leftarrow U_1(\lambda) + \Delta U_1$, and $\Omega(\lambda) \leftarrow \Omega(\lambda) + \Delta\Omega$.

4e) Compute $[\hat{Q}(\lambda), \hat{P}(\lambda)]$ from $[W_1(\lambda), U_1(\lambda), \Omega(\lambda)]$ using Eqs. (27) and (28).

4f) Recompute the errors $(E_P^*, E_Q^*, E_{\hat{P}}^*, E_{\hat{Q}}^*)$ in the correction equations (A34–A37). If $\max(\|E_P^*\|_A/\|R_1(\lambda)\|_A, \|E_Q^*\|_A/\|V_1(\lambda)\|_A, \|E_{\hat{P}}^*\|_A/\|\Sigma(\lambda)\|_A, \|E_{\hat{Q}}^*\|_A/\|\bar{\Sigma}(\lambda)\|_A) < \delta^*$, where δ^* is some preassigned correction tolerance, then set $\lambda_0 = \lambda$, and adjust next step size $\Delta\lambda$ according to the number of the correction steps required to converge before going to step 3b. Else, if the number of corrections exceeds a preset limit, reduce $\Delta\lambda$ and go to step 3b; otherwise, go to step 4b.

Step 5) If $\lambda = 1$, then stop. Else, go to step 2.

Note that the algorithm described allows the step size $(\Delta\lambda)$ to vary dynamically depending on the speed of convergence, which is gauged by the number of the correction steps. If the number is small (e.g., ≤ 3), we increase (e.g., double) the previous step size when computing the next step. If it takes many steps to converge (e.g., > 10), or does not converge, the step size is reduced (e.g., in half). The tolerance δ^* in step 4f is a preassigned correction error tolerance that can be assigned with two values in the program. One is the intermediate correction error tolerance, which is used when $\lambda < 1$. The other value is the final correction error tolerance that is usually smaller and is used when $\lambda = 1$. The choice of the magnitudes of these tolerances are problem dependent. In general, the intermediate correction tolerance is desired to be reasonably large to speed the homotopy curve following. The algorithm may fail to converge, however, if these tolerances are too large. The final correction tolerance is usually small to ensure the accuracy of the final results.

V. Illustrative Numerical Examples

In this section we present two illustrative numerical examples that demonstrate the effectiveness of the proposed algorithm. For

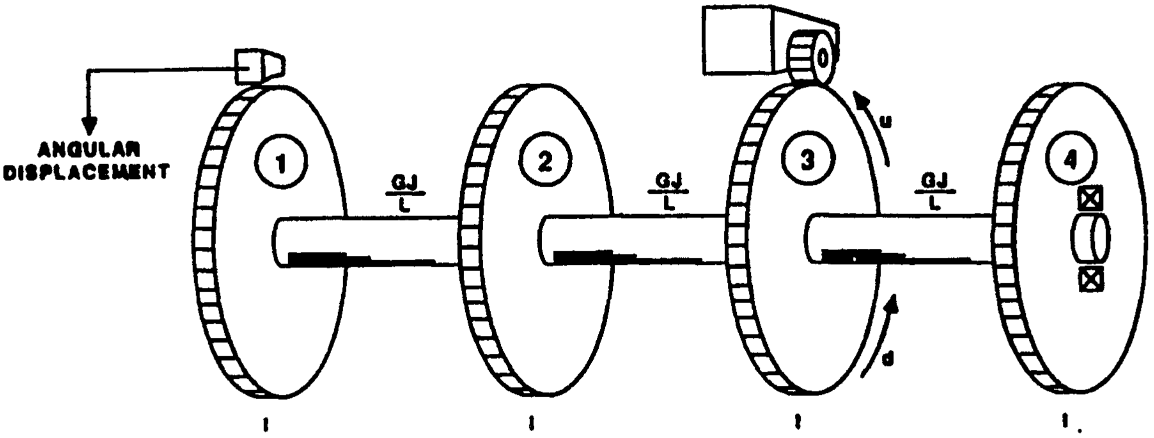

Fig. 1 Four-disk model: $I = 1$ and $GJ/L = 1$.

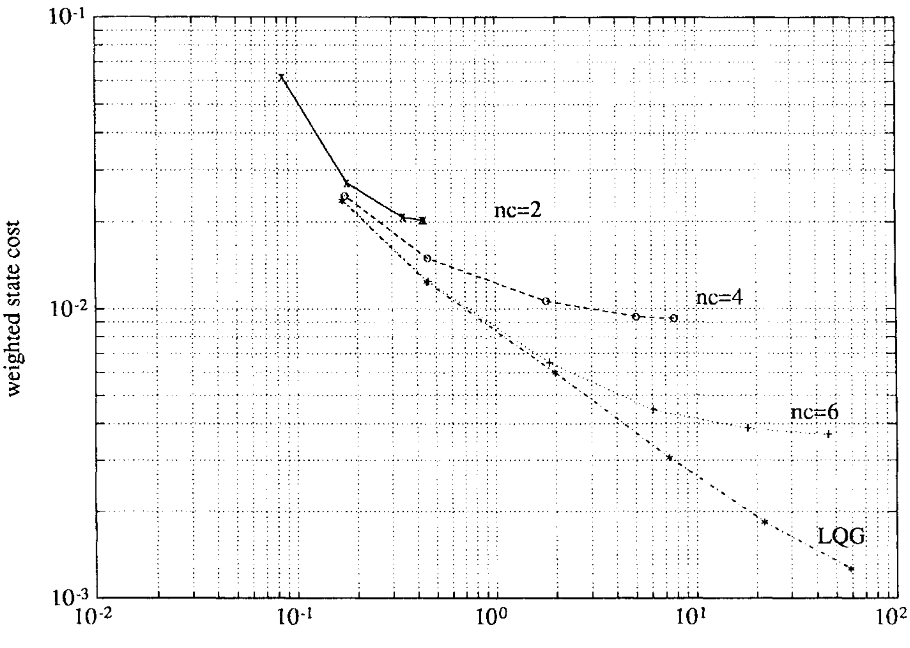

Fig. 2 Comparison of the performance curves for various order controllers for four-disk example.

both examples, the MATLAB™ implementation of the homotopy algorithm to design the optimal reduced-order compensator was run on a 486, 33-MHz personal computer. The design parameters R_2 and V_2 were allowed to vary during the homotopy path.

First, we consider a control design for an axial beam with four disks attached, as shown in Fig. 1. This example was derived from a laboratory experiment[52] and has been considered in several subsequent publications.[49,53–55] The basic control objective for the four-disk problem is to control the angular displacement at the location of disk 1 using a torque input at the location of disk 3. It is also assumed that a torque disturbance enters the system at the location of disk 3.

The design philosophy adopted here is that the scaling q_2 of the nominal control weight $R_{2,0} = 1$ and the nominal sensor noise intensity $V_{2,0} = 1$ are simply design knobs used to determine the control authority. [Hence, $R_2(\lambda) = q_2(\lambda)R_{2,0}$ and $V_2(\lambda) = q_2(\lambda)V_{2,0}$.] Here, we consider the design of second-, fourth-, and sixth-order controllers for various authority levels.

Since at $q_2 = 10$, the second-, fourth-, and sixth-reduced-order controllers obtained by balancing are all good approximations of the corresponding optimal controller, respectively, we use this suboptimal controller to initialize the homotopy algorithm and deform this controller into the higher authority optimal controller corresponding to $q_2 = 1$. In each of the following passes, we increase the authority level by decreasing R_2 and V_2 by a factor of 10, i.e., $q_{2f} = 0.1q_{2,0}$, and at the end of each pass deform the initial optimal controller to the optimal controller corresponding to the higher authority level. This process is repeated for every reduced-order design. As shown in Ref. 15, in each pass, the optimal reduced-order controller performs better than the balanced controller as the authority is increased. Figure 2 compares the optimal controllers of various orders at different authority levels. This type of figure can be used in practice to determine the order of the controller to be implemented.

Table 1 Run-time statistics of four disk example for $n_c = 4$

Control authority $q_2(1)$	MegaFLOPs	Real time, s	Predictions and corrections
10^{-1}	412	672	35
10^{-2}	407	727	35
10^{-3}	393	723	34
10^{-4}	274	478	24
10^{-5}	*	2990	120

Table 1 shows some of the run-time statistics for solving the fourth-order optimal compensator for this example at various control authorities. The cubic polynomial prediction option and the diagonal basis option were chosen for solving the coupled Lyapunov

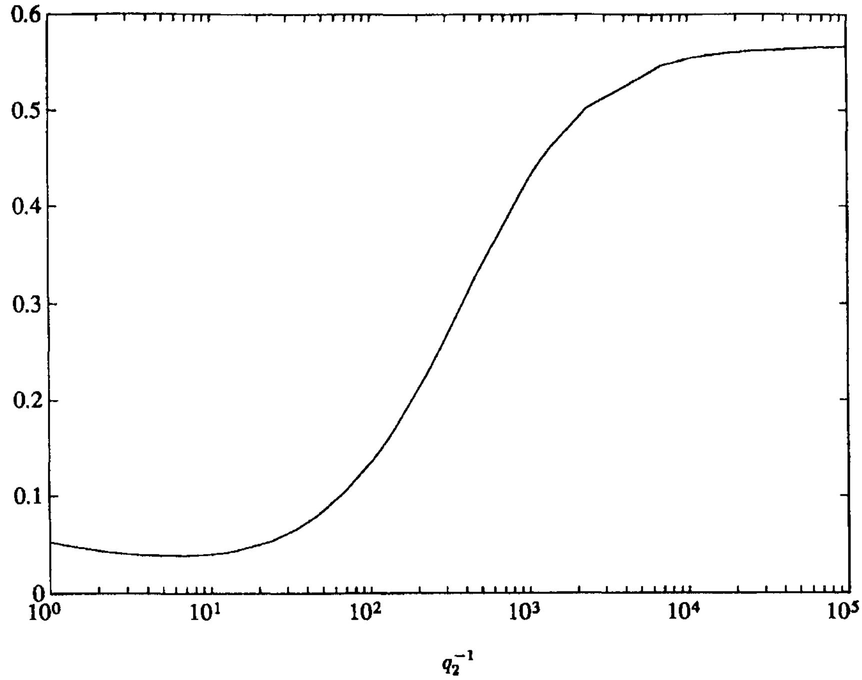

Fig. 3 $\|\hat{P}\|_F$ **as a function of control authority** q_2^{-1} **for four-disk example with** $n_c = 4$.

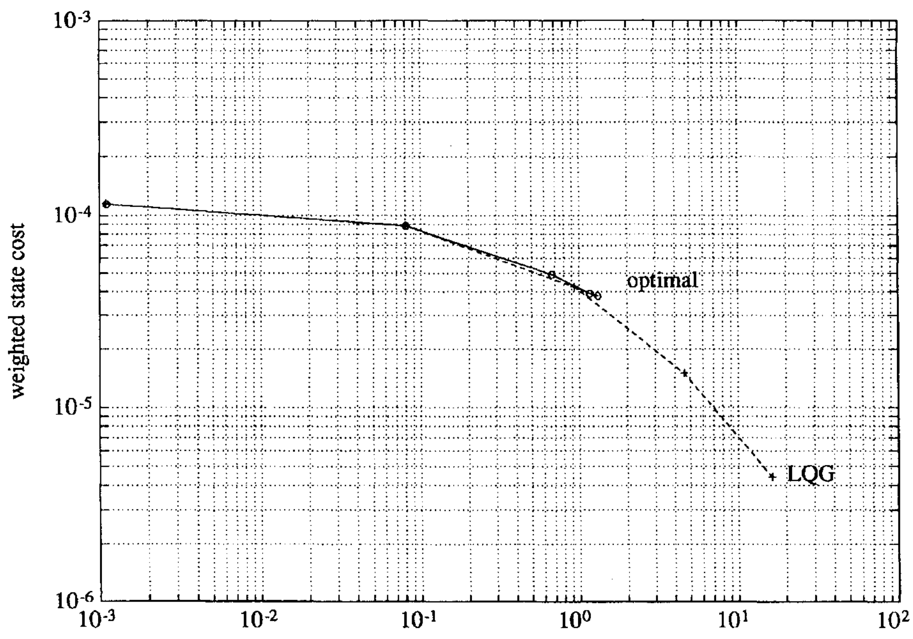

Fig. 4 Performance curves for the eighth-order controllers for ACES structure.

equations in this comparison. For $q_2(1) = 10^{-5}$, however, diagonalizing errors are significant, and the basis option was switched to the Schur form. The asterisk under the MegaFLOPs heading indicates that the MATLAB FLOP counter overflowed and so the FLOP data are unavailable. Standard descent and gradient-based continuation algorithms often failed on this problem when the controller was constrained to a particular basis. When these methods worked, however, the convergence time was improved by an order of magnitude over the convergence times recorded in Table 1. These results are further discussed in Ref. 56.

The Frobenius norms of P, Q, $\hat{P}$, and $\hat{Q}$ are also recorded along the homotopy path. A typical curve of $\|\hat{P}\|_F$ is shown in Fig. 3 for the fourth-order controller design. The Frobenius norms of P, Q, and $\hat{Q}$ exhibit similar characteristics. Note that as the control authority is increased beyond a certain level (e.g., for $n_c = 4$ and $q_2 < 10^{-4}$) those values approach some stable limit, as indicated in the figure. This is because P, Q, $\hat{P}$, and $\hat{Q}$ converge to fixed values as the control authority increases. It follows that the optimal reduced-order controller converges to a fixed value. This later phenomenon has been observed previously.[15] Furthermore, most of the prediction/correction steps indicated in Table 1 for this special case $[q_2(1) = 10^{-5}]$ occur when λ is approaching 1 and P, Q, $\hat{P}$, and $\hat{Q}$ approach their final fixed values.

Note that the described phenomenon contrasts with the full-order case in which convergence as q_2 decreases never occurs. This is because for full-order control the infimum of the output cost as a function of the input cost is unattainable, whereas for reduced-order control the infimum of the output cost is actually attainable.

Algorithm efficiency as a function of the prediction options and basis options for solving the coupled Lyapunov equations has been studied in the context of a similar algorithm for H_2 optimal model reduction using the corresponding optimal projection equations.[24] It was seen that the algorithm is most efficient when using the cubic polynomial prediction and diagonalizing the coefficient matrices of the coupled Lyapunov equations. These conclusions also hold for the algorithm presented here.

The second example illustrates the design of an optimal reduced-order controller for a 17th-order model of one of the single-input, single-output transfer functions of the NASA Marshall Active Control Technique Evaluation for Spacecraft structure.[14,57] The actuator and sensor are a torque actuator and a collocated rate gyro, respectively. The model includes the actuator and sensor dynamics. A first-order all-pass filter was appended to the model to approximate the computational delay associated with digital implementation.

Following the same approach, we design an eighth-order controller for this plant. Figure 4 shows the performance curves for authority levels corresponding to $q_2 \in (10^{-3}, 10^{-4}, \ldots, 10^{-7})$ and compares the optimal curves for an LQG controller with the optimal reduced-order controller. For this special case, suboptimal reduced-order controllers obtained by balancing destabilize the closed-loop system when $q_2 \leq 10^{-5}$.

The run time statistics for solving the eighth-order optimal compensator for this example at various control authorities are as follows: $q_2(1) = 10^{-4} \to 4098$ s, $q_2(1) = 10^{-5} \to 9008$ s, $q_2(1) = 10^{-6} \to 4712$ s, $q_2(1) = 10^{-7} \to 2216$ s. Note that the MATLAB FLOP counter overflowed for each of these cases, whereas the number of predictions/corrections were 19, 42, 22, and 10, respectively. The cubic polynomial prediction option and the diagonal basis option were chosen for solving the coupled Lyapunov equations in this comparison.

VI. Conclusion

Gradient-based minimal parameterization homotopy algorithms for H_2 optimal reduced-order dynamic compensation[15] are computationally efficient in theory, but tend to experience numerical ill conditioning in practice resulting from the constraint on the controller basis. Hence, this paper has presented a new homotopy algorithm for the synthesis of H_2 optimal reduced-order compensators based on directly solving the optimal projection equations that characterize the optimal compensator. The resulting algorithm is usually more numerically robust than the gradient-based homotopy algorithms. The number of variables associated with this approach is $(2n + n_c)(m + l)$, however, which is greater than the number of variables associated with minimal parameterization approach $n_c(m + l)$. The two examples of the preceding section demonstrate the effectiveness of the proposed algorithm.

Appendix: Formulation of the Derivative and Correction Equations

Before deriving the derivative and correction equations (35–42), we state the following useful properties about the derivatives of the contragredient transformation of $(\hat{Q}, \hat{P})$.

Note that it follows from Eqs. (27–29) that $\tau(\lambda)$ can be expressed as

$$\tau(\lambda) = \hat{Q}(\lambda)U^T(\lambda)\Lambda^{\dagger}(\lambda)U(\lambda) = \hat{Q}(\lambda)U_1(\lambda)\Omega^{-1}(\lambda)U_1^T(\lambda) \tag{A1}$$

or, equivalently,

$$\tau(\lambda) = W(\lambda)\Lambda^{\dagger}(\lambda)W^T(\lambda)\hat{P}(\lambda = W_1(\lambda)\Omega^{-1}(\lambda)W_1^T(\lambda)\hat{P}(\lambda) \tag{A2}$$

where

$$\Lambda^{\dagger}(\lambda) = \begin{bmatrix} \Omega^{-1}(\lambda) & 0 \\ 0 & 0 \end{bmatrix} \tag{A3}$$

The representations of $\tau(\lambda)$ given by Eqs. (A1) and (A2) are used next as a convenient way of expressing the derivative equations partially in terms of $\dot{\hat{Q}}(\lambda)$ and $\dot{\hat{P}}(\lambda)$ as opposed to expressing the derivative equations only in terms of $\dot{W}_1(\lambda)$, $\dot{U}_1(\lambda)$, and $\dot{\Omega}(\lambda)$. For notational simplicity, we omit the argument λ in the subsequent equations.

Differentiating Eq. (A1) or (A2) gives the following expressions for $\dot{\tau}$:

$$\dot{\tau} = \dot{\hat{Q}}U_1\Omega^{-1}U_1^T + \hat{Q}\big(\dot{U}_1\Omega^{-1}U_1^T + U_1\Omega^{-1}\dot{\Omega}\Omega^{-1}U_1^T + U_1\Omega^{-1}\dot{U}_1^T\big) \tag{A4}$$

or, equivalently,

$$\dot{\tau} = W_1\Omega^{-1}W_1^T\dot{\hat{P}} + \big(W_1\Omega^{-1}\dot{W}_1^T + W_1\Omega^{-1}\dot{\Omega}\Omega^{-1}W_1^T + \dot{W}_1\Omega^{-1}W_1^T\big)\hat{P} \tag{A5}$$

with

$$\frac{\mathrm{d}\Omega^{-1}}{\mathrm{d}\lambda} = -[\Omega^{-1}]^2\dot{\Omega} = -\Omega^{-1}\dot{\Omega}\Omega^{-1} \tag{A6}$$

since Ω is diagonal. Next, we derive the matrix equations that can be used to solve for the derivatives and corrections.

Derivative Equations

Differentiating Eqs. (21–24) with respect to λ and using Eqs. (A4–A6) yields Eqs. (35–38) with

$$A_P \triangleq A - \Sigma P \tau \tag{A7}$$

$$A_w \triangleq A - Q\bar{\Sigma} + W_1\Omega^{-1}W_1^T P\Sigma P(I_n - \tau) \tag{A8}$$

$$A_Q \triangleq A - \tau Q\bar{\Sigma} \tag{A9}$$

$$A_u \triangleq A - \Sigma P + (I_n - \tau)Q\bar{\Sigma}QU_1\Omega^{-1}U_1^T \tag{A10}$$

$$R_P(\dot{P}, \dot{Q}, \dot{\hat{P}}, \dot{\hat{Q}}) \triangleq -\hat{P}\alpha(\dot{\hat{P}}, \dot{\hat{Q}})P\Sigma P(I_n - \tau) - (I_n - \tau)^T P\Sigma\dot{P}\tau - \dot{\hat{P}}W_1\Omega^{-1}W_1^T P\Sigma P(I_n - \tau) \tag{A11}$$

$$R_Q(\dot{P}, \dot{Q}, \dot{\hat{P}}, \dot{\hat{Q}}) \triangleq -\hat{Q}\gamma(\dot{\hat{P}}, \dot{\hat{Q}})Q\bar{\Sigma}Q(I_n - \tau)^T - \tau\dot{Q}\bar{\Sigma}Q(I_n - \tau)^T - \dot{\hat{Q}}U_1\Omega^{-1}U_1^T Q\bar{\Sigma}Q(I_n - \tau)^T \tag{A12}$$

$$R_{\hat{P}}(\dot{P}, \dot{Q}, \dot{\hat{P}}, \dot{\hat{Q}}) \triangleq \hat{P}\alpha(\dot{\hat{P}}, \dot{\hat{Q}})P\Sigma P(I_n - \tau) + (I_n - \tau)^T P\Sigma\dot{P}\tau + \tau^T P\Sigma\dot{P} - \bar{\Sigma}\dot{Q}\hat{P} \tag{A13}$$

$$R_{\hat{Q}}(\dot{P}, \dot{Q}, \dot{\hat{P}}, \dot{\hat{Q}}) \triangleq \hat{Q}\gamma(\dot{\hat{P}}, \dot{\hat{Q}})Q\bar{\Sigma}Q(I_n - \tau)^T + \tau\dot{Q}\bar{\Sigma}Q(I_n - \tau)^T + \tau Q\bar{\Sigma}\dot{Q} - \Sigma\dot{P}\hat{Q} \tag{A14}$$

$$\alpha(\dot{\hat{P}}, \dot{\hat{Q}}) \triangleq \dot{W}_1\Omega^{-1}W_1^T - W_1\Omega^{-1}\dot{\Omega}\Omega^{-1}W_1^T + W_1\Omega^{-1}\dot{W}_1^T \tag{A15}$$

$$\gamma(\dot{\hat{P}}, \dot{\hat{Q}}) \triangleq \dot{U}_1\Omega^{-1}U_1^T - U_1\Omega^{-1}\dot{\Omega}\Omega^{-1}U_1^T + U_1\Omega^{-1}\dot{U}_1^T \tag{A16}$$

$$V_P \triangleq \dot{A}^T P - \big(I_n - \tfrac{1}{2}\tau\big)^T P\dot{\Sigma}P\tau \tag{A17}$$

$$V_Q \triangleq \dot{A}Q - \big(I_n - \tfrac{1}{2}\tau\big)Q\dot{\bar{\Sigma}}Q\tau^T \tag{A18}$$

$$V_{\hat{P}} \triangleq (\dot{A} + Q\dot{\bar{\Sigma}})^T\hat{P} + \big(I_n - \tfrac{1}{2}\tau\big)^T P\dot{\Sigma}P\tau \tag{A19}$$

$$V_{\hat{Q}} \triangleq (\dot{A} - \dot{\Sigma}P)\hat{Q} + \big(I_n - \tfrac{1}{2}\tau\big)Q\dot{\bar{\Sigma}}Q\tau^T \tag{A20}$$

Note that it follows from Eqs. (51–53) that

$$\dot{A} = A_f - A_0, \qquad \dot{B} = B_f - B_0, \qquad \dot{C} = C_f - C_0 \tag{A21}$$

$$\dot{R}_1 = R_{1,f} - R_{1,0}, \qquad \dot{R}_2 = R_{2,f} - R_{2,0} \tag{A22}$$

$$\dot{V}_1 = V_{1,f} - V_{1,0}, \qquad \dot{V}_2 = V_{2,f} - V_{2,0} \tag{A23}$$

$$\dot{\Sigma} = \dot{B}R_2^{-1}B^T - BR_2^{-2}\dot{R}_2B^T + BR_2^{-1}\dot{B}^T, \qquad \dot{\bar{\Sigma}} = \dot{C}^T V_2^{-1}C + C^T V_2^{-2}\dot{V}_2C + C^T V_2^{-1}\dot{C} \tag{A24}$$

Next, differentiating Eqs. (27) and (28) yields

$$\dot{\hat{Q}} = \dot{W}_1 \Omega W_1^T + W_1 \dot{\Omega} W_1^T + W_1 \Omega \dot{W}_1^T \tag{A25}$$

$$\dot{\hat{P}} = \dot{U}_1 \Omega U_1^T + U_1 \dot{\Omega} U_1^T + U_1 \Omega \dot{U}_1^T \tag{A26}$$

Furthermore, differentiating Eq. (73) with respect to λ gives

$$\dot{U} W + U \dot{W} = \dot{U}_1^T W_1 + U_1^T \dot{W}_1 = 0 \tag{A27}$$

Correction Equations

The correction equations are developed with λ at some fixed value, say, λ^*. The derivation of the correction equations are based on the relationship between Newton's method and a particular homotopy. Next, we use the notation

$$\dot{f}(\theta) \triangleq \frac{\partial f}{\partial \theta} \tag{A28}$$

Let $f: \boldsymbol{R}^n \to \boldsymbol{R}^n$ be C^1 continuous and consider the equation

$$f(\theta) = 0 \tag{A29}$$

If $\theta^{(i)}$ is the current approximation to the solution of Eq. (A29), then the Newton correction[57] $\Delta\theta$ is given by

$$\theta^{(i+1)} - \theta^{(i)} \triangleq \Delta\theta = -\dot{f}\left(\theta^{(i)}\right)^{-1} e \tag{A30}$$

where

$$e \triangleq f\left(\theta^{(i)}\right) \tag{A31}$$

Now, let $\theta^{(i)}$ be an approximation to θ satisfying Eq. (A29). Then with e given by Eq. (A31) construct the following homotopy to solve Eq. (A29):

$$(1-\beta)e = f[\theta(\beta)], \qquad \beta \in [0, 1] \tag{A32}$$

Note that at $\beta = 0$ Eq. (A32) has solution $\theta(0) = \theta^{(i)}$, whereas $\theta(1)$ satisfies Eq. (A29). Then differentiating Eq. (A32) with respect to β gives

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}\beta}\right|_{\beta=0} = -\dot{f}\left(\theta^{(i)}\right)^{-1} e \tag{A33}$$

Remark A.1. Note that the Newton correction $\Delta\theta$ in Eq. (A30) and the derivative $(\mathrm{d}\theta/\mathrm{d}\beta)|_{\beta=0}$ in Eq. (A33) are identical. Hence, the Newton correction $\Delta\theta$ can be found by constructing a homotopy of the form (A32) and solving for the resulting derivative $(\mathrm{d}\theta/\mathrm{d}\beta)|_{\beta=0}$. As will be seen, this insight is particularly useful when deriving Newton corrections for equations that have a matrix structure.

Now, we use the insights of Remark A.1 to derive the equations that need to be solved for the Newton corrections (ΔP, ΔQ, $\Delta\hat{P}$, $\Delta\hat{Q}$) or, equivalently, (ΔP, ΔQ, ΔW_1, ΔU_1, $\Delta\Omega$). We begin by recalling that λ is assumed to have some fixed value, say, λ^*. Also, it is assumed that (P^*, Q^*, $\hat{P}^*$, $\hat{Q}^*$) is the current approximation of $[P(\lambda^*), Q(\lambda^*), \hat{P}(\lambda^*), \hat{Q}(\lambda^*)]$ and that (E_P^*, E_Q^*, $E_{\hat{P}}^*$, $E_{\hat{Q}}^*$) are, respectively, the errors in Eqs. (21–24) with $\lambda = \lambda^*$ and $[P(\lambda), Q(\lambda), \hat{P}(\lambda), \hat{Q}(\lambda)]$ replaced by (P^*, Q^*, $\hat{P}^*$, $\hat{Q}^*$), respectively.

Next, we form the homotopy

$$\begin{aligned}(1-\beta)E_P^* &= A^T P(\beta) + P(\beta)A + R_1 \\ &+ \tau^T(\beta)P(\beta)\Sigma P(\beta)\tau(\beta) - \tau^T(\beta)P(\beta)\Sigma P(\beta) \\ &- P(\beta)\Sigma P(\beta)\tau(\beta)\end{aligned} \tag{A34}$$

$$\begin{aligned}(1-\beta)E_Q^* &= AQ(\beta) + Q(\beta)A^T + V_1 \\ &+ \tau(\beta)Q(\beta)\bar{\Sigma}Q(\beta)\tau^T(\beta) - \tau(\beta)Q(\beta)\bar{\Sigma}Q(\beta) \\ &- Q(\beta)\bar{\Sigma}Q(\beta)\tau^T(\beta)\end{aligned} \tag{A35}$$

$$\begin{aligned}(1-\beta)E_{\hat{P}}^* &= [A - Q(\beta)\bar{\Sigma}]^T \hat{P}(\beta) + \hat{P}(\beta)[A - Q(\beta)\bar{\Sigma}] \\ &- \tau^T(\beta)P(\beta)\Sigma P(\beta)\tau(\beta) + \tau^T(\beta)P(\beta)\Sigma P(\beta) \\ &+ P(\beta)\Sigma P(\beta)\tau(\beta)\end{aligned} \tag{A36}$$

$$\begin{aligned}(1-\beta)E_{\hat{Q}}^* &= [A - \Sigma P(\beta)]\hat{Q}(\beta) + \hat{Q}(\beta) \\ &\times [A - \Sigma P(\beta)]^T - \tau(\beta)Q(\beta)\bar{\Sigma}Q(\beta)\tau^T(\beta) \\ &+ \tau(\beta)Q(\beta)\bar{\Sigma}Q(\beta) + Q(\beta)\bar{\Sigma}Q(\beta)\tau^T(\beta)\end{aligned} \tag{A37}$$

Here, $(A, \Sigma, \bar{\Sigma}, R_1, V_1) = [A(\lambda^*), \Sigma(\lambda^*), \bar{\Sigma}(\lambda^*), R_1(\lambda^*), V_1(\lambda^*)]$ are assumed to be evaluated at $\lambda = \lambda^*$ and at $\beta = 0$, $[P(0), Q(0), \hat{P}(0), \hat{Q}(0), \tau(0)]$ are the current approximations. Differentiating Eqs. (A34–A37) with respect to β, noting the identity of Eqs. (A4–A6) with $\dot{\tau}$ now representing $\mathrm{d}\tau/\mathrm{d}\beta$, and using Remark A.1 to make the replacements

$$\Delta P \triangleq \left.\frac{\mathrm{d}P}{\mathrm{d}\beta}\right|_{\beta=0}, \qquad \Delta Q \triangleq \left.\frac{\mathrm{d}Q}{\mathrm{d}\beta}\right|_{\beta=0}$$
$$\Delta \hat{P} \triangleq \left.\frac{\mathrm{d}\hat{P}}{\mathrm{d}\beta}\right|_{\beta=0}, \qquad \Delta \hat{Q} \triangleq \left.\frac{\mathrm{d}\hat{Q}}{\mathrm{d}\beta}\right|_{\beta=0} \tag{A38}$$

yields Eqs. (39–42), where A_P, A_w, A_Q, and A_u are defined in Eqs. (A7–A10), respectively, and

$$\begin{aligned}R_P(\Delta P, \Delta Q, \Delta\hat{P}, \Delta\hat{Q}) &\triangleq -\hat{P}\alpha(\Delta\hat{P}, \Delta\hat{Q})P\Sigma P(I_n - \tau) \\ &- (I_n - \tau)^T P\Sigma\Delta P\tau - \Delta\hat{P}W_1\Omega^{-1}W_1^T P\Sigma P(I_n - \tau)\end{aligned} \tag{A39}$$

$$\begin{aligned}R_Q(\Delta P, \Delta Q, \Delta\hat{P}, \Delta\hat{Q}) &\triangleq -\hat{Q}\gamma(\Delta\hat{P}, \Delta\hat{Q})Q\bar{\Sigma}Q(I_n - \tau)^T \\ &- \tau\Delta Q\bar{\Sigma}Q(I_n - \tau)^T - \Delta\hat{Q}U_1\Omega^{-1}U_1^T Q\bar{\Sigma}Q(I_n - \tau)^T\end{aligned} \tag{A40}$$

$$\begin{aligned}R_{\hat{P}}(\Delta P, \Delta Q, \Delta\hat{P}, \Delta\hat{Q}) &\triangleq \hat{P}\alpha(\Delta\hat{P}, \Delta\hat{Q})P\Sigma P(I_n - \tau) \\ &+ (I_n - \tau)^T P\Sigma\Delta P\tau + \tau^T P\Sigma\Delta P - \bar{\Sigma}\Delta Q\hat{P}\end{aligned} \tag{A41}$$

$$\begin{aligned}R_{\hat{Q}}(\Delta P, \Delta Q, \Delta\hat{P}, \Delta\hat{Q}) &\triangleq \hat{Q}\gamma(\Delta\hat{P}, \Delta\hat{Q})Q\bar{\Sigma}Q(I_n - \tau)^T \\ &+ \tau\Delta Q\bar{\Sigma}Q(I_n - \tau)^T + \tau Q\bar{\Sigma}\Delta Q - \Sigma\Delta P\hat{Q}\end{aligned} \tag{A42}$$

$$\begin{aligned}\alpha(\Delta\hat{P}, \Delta\hat{Q}) &\triangleq \Delta W_1\Omega^{-1}W_1^T - W_1\Omega^{-1}\Delta\Omega\Omega^{-1}W_1^T \\ &+ W_1\Omega^{-1}\Delta W_1^T W_1^T\end{aligned} \tag{A43}$$

$$\begin{aligned}\gamma(\Delta\hat{P}, \Delta\hat{Q}) &\triangleq \Delta U_1\Omega^{-1}U_1^T \\ &- U_1\Omega^{-1}\Delta\Omega\Omega^{-1}U_1^T + U_1\Omega^{-1}\Delta U_1^T\end{aligned} \tag{A44}$$

Next, replacing λ with β in Eqs. (27), (28), and (33) and differentiating with respect to β yields

$$\Delta\hat{Q} = \Delta W_1\Omega W_1^T + W_1\Delta\Omega W_1^T + W_1\Omega\Delta W_1^T \tag{A45}$$

$$\Delta\hat{P} = \Delta U_1\Omega U_1^T + U_1\Delta\Omega U_1^T + U_1\Omega\Delta U_1^T \tag{A46}$$

$$\Delta U_1^T W_1 + U_1^T \Delta W_1 = 0 \tag{A47}$$

where

$$\Delta U_1 \triangleq \left.\frac{\mathrm{d}U}{\mathrm{d}\beta}\right|_{\beta=0}, \qquad \Delta W_1 \triangleq \left.\frac{\mathrm{d}W}{\mathrm{d}\beta}\right|_{\beta=0}, \qquad \Delta\Omega_1 \triangleq \left.\frac{\mathrm{d}\Omega}{\mathrm{d}\beta}\right|_{\beta=0}$$

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